Continuum state cellular automata description using random differential equations

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In the following working paper we prove that the state variable of our CA verify a suitable type of RDE. The notion of derivative of stochastic process we will use is the forward mean derivative

$$y'(t) := \lim_{\Delta t \to 0+} E_t\left(\frac{y_{t+\Delta t} - y_t}{\Delta t}\right)$$

where $E_t(-) := E(-|\mathfrak{S}_t)$ and $\mathfrak{S}_t := \sigma\left[(y_s)_{s \le t}\right]$ is the sigma algebra generated by present and past states of y (it is well known that $\mathfrak{S}_t$ can be interpreted as the set of all the information expressible using the process $(y_s)_{s \le t}$ and hence this is a possible way to consider memory effect). For this reason every probability that appears in this document is to be meant as $P_t := P[-|\mathfrak{S}_t]$.

In this proof we used a rigorous theory of actual infinitesimals (see [Gio 04]) to simplify the calculations; this theory permits to write exact equalities instead of approximations, e.g. $f(x+dx) = f(x) + dx \cdot f'(x)$ if dx is a first order nilpotent infinitesimal, that is $dx \neq 0$ but $(dx)^2 = 0$, or $\int_x^{x+dx} f(t)\,\mathrm{d}t = f(x) \cdot dx$ for the same type of infinitesimal.

The RDE for the (conditional) mean value (first moment)

First of all we start remembering the synchronous version of the algorithm used for the CA (due to the well know relations between Poisson and exponential distribution, the synchronous and the asynchronous version of the algorithm are mathematically (not numerically) equivalent). For more details about the algorithm see [Van et al 04].

We proceed cell by cell and at a fixed time t. To simplify the notations sometimes we suppress the dependence of some variables by the time t or by the cell c.

1. The initial step $t = 0$ is given by a deterministic global configuration of the system.
2. First of all we have to calculate, using the present value $e_t \in E = \mathbb{R}^D$ of the state variables of the cell and its neighbourhoods, the total intensity $\lambda_c := \sum_{\alpha \in A} \lambda_{\alpha c}$ of the sum process in the cell c, where we have used the simplified notation $\lambda_{\alpha c} := \lambda_\alpha(c)$. The sum process can be roughly described as "an event of type α, for some α, happens in the cell c".
3. Using the total intensity, the number of events k, of some type ,α is extracted using a Poisson distribution

   $$P_t[N_c = k] = \frac{(\lambda_c \cdot \Delta t)^k}{k!} \cdot \exp(-\lambda_c \cdot \Delta t)$$

   Here $N_c = N_c(t)$ is the random variable which counts the number of events happened in the cell.

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4. Having k, using a multinomial distribution we can calculate the number of events N_α of type α which happen in the cell for each α.

   So if $k = \sum_{\alpha \in A} k_\alpha$ and $p_{\alpha c} := \frac{\lambda_{\alpha c}}{\lambda_c}$, then

   $$P_t\left[N_{\alpha c} = k_\alpha \quad \forall \alpha \in A \,\middle|\, N_c = k \right] = \frac{k!}{\prod_{\alpha \in A} k_\alpha!} \cdot \prod_{\alpha \in A} \left(p_{\alpha c} \right)^{k_\alpha} . \tag{1.1}$$

   Note that this is a general formula, but concretely $N_\alpha(t) \in \{0,1\}$ because of the urban meaning of each α-transformation: e.g. if α = "Construction of a residential building on a free terrain" then only one builder has got the property of the terrain and so only one α-event may at least happen. Anyway the following proof is written using the general formula (1.1) and so it does not depend on this peculiar characteristic of our model.

5. At this step of the algorithm we need to extract, using their probability density defined in our context using fuzzy logic methods, the value of the stochastic goods $\pi \in \mathbb{R}^{n(\alpha)}$ associated to each α-transformation. This step is formalized by the following equality

   $$P_t\left[\left(\Pi_1^\alpha, ..., \Pi_{n(\alpha)}^\alpha \right) \in B \,\middle|\, N_c = k, N_{\alpha c} = k_\alpha \quad \forall \alpha \in A \right] = \int_B \beta_\alpha(c, \pi) \cdot \mathrm{d}\pi \text{ if } k = \sum_{\alpha \in A} k_\alpha \tag{1.2}$$

   where $B \in \mathcal{B}\left(\mathbb{R}^{n(\alpha)} \right)$ is a measurable set, $\Pi^\alpha(t) := \left(\Pi_1^\alpha(t), ..., \Pi_{n(\alpha)}^\alpha(t) \right)$ is the vector random variable describing the goods produced by the α-transformation and $\beta_\alpha(c, \pi) \cdot \mathrm{d}\pi = \beta_\alpha\left(c, \pi_1, ..., \pi_{n(\alpha)} \right) \cdot \mathrm{d}\pi_1 \cdot ... \cdot \mathrm{d}\pi_{n(\alpha)}$ is the above mentioned probability density. This step has to be performed for each one of the k_α events and for every $\alpha \in A$.

6. If $\pi_1^\alpha, ..., \pi_{n(\alpha)}^\alpha$ are the values of Π^α extracted at the previous step, then we update each state variable of the cell c using a relation like the (3.3.3) of [Van et al 04]. We remember here the equality (3.3.3) in a more general form

   $$v_{11}(c, t + \Delta t) = v_{11}(c, t) + \sum_{\alpha=7}^{10} \sum_{k=1}^{k_\alpha} \left[\gamma_\alpha^{11}\left(\pi_1^\alpha, ..., \pi_{n(\alpha)}^\alpha \right) - \gamma_{11}^{11}\left(\pi_1^{11} \right) \right] \tag{1.3}$$

   here $k_\alpha = 0$ if no α-transformation happen in the cell (and hence $v_{11}(c, t + \Delta t) = v_{11}(c, t)$), and $k_\alpha = 1$ if an α-transformation happened in the cell (k_α are the values of the random variable N_α introduced at step 3).

This completes the algorithm. To prove the first RDE we start writing the equality (1.3) (which is written using the extracted values π^α of the random variable Π^α and k_α of the variable N_α) as

$$v_{11}(c, t + \Delta t) = v_{11}(c, t) + \sum_{\alpha=7}^{10} \sum_{k=1}^{N_\alpha(t)} \left[\gamma_\alpha^{11}\left(\Pi_1^\alpha(t), ..., \Pi_{n(\alpha)}^\alpha(t) \right) - \gamma_{11}^{11}\left(\Pi_1^{11}(t) \right) \right] \tag{1.4}$$

This can be thought as the definition of the stochastic process $v_{11}(c, t)$ at the next time step. To simplify a little bit the notation we call

$S(c,t) \coloneqq S_t \coloneqq v_{11}(c,t)$,
and

$$S_\alpha(t) \coloneqq \gamma_\alpha^{11}\left(\Pi^\alpha(t)\right) - \gamma_{11}^{11}\left(\Pi_1^{11}(t)\right), \tag{1.5}$$

so that now (1.4) can be written as

$$S_{t+\Delta t} = S_t + \sum_{\alpha=7}^{10} \sum_{k=1}^{N_\alpha(t)} S_\alpha(t). \tag{1.6}$$

Note that because of the simple affine functions $\gamma_\alpha = \left(\gamma_\alpha^1,...,\gamma_\alpha^{11}\right)$ (see equation (3.3.2) in [Van et al 04]) from (1.5) we can simply obtain the probability density of S_α conditioned to a given value π of Π^α and of N_c, N_α from the densities of Π^α. We will indicate the density of S_α with $\beta_{11}\left(s;k,(k_\alpha)_{\alpha\in A},\pi\right)$[5].
Taking the (conditional expected) value of both terms of (1.6) we obtain

$$E_t\left(S_{t+\Delta t}\right) = E_t\left(S_t\right) + \sum_{\alpha=7}^{10} E_t\left[\sum_{k=1}^{N_\alpha(t)} S_\alpha(t)\right]. \tag{1.7}$$

Note that we cannot use the linearity of E_t for the second sum because $N_\alpha(t)$ is a random variable.
Now we calculate the expected value of this sum as:

$$E_t\left[\sum_{k=1}^{N_\alpha(t)} S_\alpha\right] = \int_B E_t\left[\sum_{k=1}^{N_\alpha(t)} S_\alpha \,\middle|\, N_c = k, N_\beta = k_\beta \;\forall\beta\in A, \Pi^\alpha = \pi\right]\cdot... \\ ...\cdot P_t\left[N_c = k, N_\beta = k_\beta \;\forall\beta\in A, \Pi^\alpha \in (\pi, \pi + \mathrm{d}\pi)\right] \tag{1.8}$$

(so now α is a fixed valued between 7 and 10), where

$$B \coloneqq \left\{\left(k,(k_\beta)_{\beta\in A},\pi\right)\in \mathbb{N}\times\mathbb{N}^A\times\mathbb{R}^{n(\alpha)} \;\middle|\; k = \sum_{\beta\in A} k_\beta\right\}$$

is the set of all the values of the variables N_c, N_β for $\beta\in A$ and Π^α. But at the step 4 of the algorithm we have seen that

$$P_t\left[\Pi^\alpha \in B \,\middle|\, N_c = k, N_\beta = k_\beta \quad \forall\beta\in A\right] = \int_B \beta_\alpha(\pi)\cdot\mathrm{d}\pi \text{ if } k = \sum_{\beta\in A} k_\beta \tag{1.9}$$

so

[5] In the following with the notation $\beta(-;y)$ (note the use of ; instead of ,) we indicate the conditional probability density of the variable X conditioned by $Y = y$.

$$P_t\left[N_c = k, N_\beta = k_\beta \ \forall \beta \in A, \Pi^\alpha \in (\pi, \pi + \mathrm{d}\pi]\right] = \beta_\alpha(\pi) \cdot \mathrm{d}\pi \cdot P_t\left[N_c = k, N_\beta = k_\beta \ \forall \beta \in A\right] \tag{1.10}$$

where we have taken in (1.9) as B the infinitesimal interval $B = (\pi, \pi + \mathrm{d}\pi]$ so that

$$\int_B \beta_\alpha(\pi) \cdot \mathrm{d}\pi = \int_\pi^{\pi + \mathrm{d}\pi} \beta_\alpha(\pi) \cdot \mathrm{d}\pi = \beta_\alpha(\pi) \cdot \mathrm{d}\pi .$$

Now take Δt a first order infinitesimal, that is $\Delta t \neq 0$ but $(\Delta t)^2 = 0$. For the well known properties of the Poisson distribution[6] it follows that

$$\begin{aligned} &P_t[N_c = 0] = 1 - \lambda_c \cdot \Delta t \\ &P_t[N_c = 1] = \lambda_c \cdot \Delta t \\ &P_t[N_c = k] = 0 \qquad \forall k \geq 2 \end{aligned} \tag{1.11}$$

Now if $k = \sum_{\beta \in A} k_\beta = 0$ then $k_\beta = 0 \ \forall \beta \in A$, and so $N_\alpha = k_\alpha = 0$ and thus $E_t\left(\sum_{k=1}^{N_\alpha(t)} S_\alpha\right) = 0$; analogously we can proceed if $k = \sum_{\beta \in A} k_\beta = 1$ but $k_\alpha = 0$. If $k \geq 2$ then from (1.11) we have that $0 = P_t[N_c = k] \geq P_t\left[N_c = k, N_\beta = k_\beta \ \forall \beta \in A\right]$. At the end only the case $N_c = k = 1 = N_\alpha = k_\alpha$ remains and the expected value simplifies as follows

$$\begin{aligned} E_t\left[\sum_{k=1}^{N_\alpha(t)} S_\alpha\right] &= \int_B E_t\left[\sum_{k=1}^{N_\alpha(t)} S_\alpha \,\middle|\, N_c = 1 = N_\alpha, \Pi^\alpha = \pi\right] \cdot \beta_\alpha(\pi) \cdot \mathrm{d}\pi \cdot P_t\left[N_c = 1 = N_\alpha\right] = \\ &= \int_{\mathbb{R}} E_t\left[S_\alpha \,\middle|\, N_c = 1 = N_\alpha, \Pi^\alpha = \pi\right] \cdot \beta_\alpha(\pi) \cdot \mathrm{d}\pi \cdot P_t\left[N_c = 1 = N_\alpha\right] \end{aligned} \tag{1.12}$$

where we used both (1.8) and (1.10) and where we did not write the sum because $N_\alpha = N_\alpha(t) = 1$. But (1.1) at the step 3 of the algorithm says

$$P\left[N_\alpha = k_\alpha \quad \forall \alpha \in A \,\middle|\, N_c = k\right] = \frac{k!}{\prod_{\alpha \in A} k_\alpha!} \cdot \prod_{\alpha \in A} (p_{\alpha c})^{k_\alpha}$$

In our case it becomes

$$P_t[N_c = 1 = N_\alpha] = p_{\alpha c} \cdot P_t[N_c = 1] = p_{\alpha c} \cdot \lambda_c \cdot \Delta t = \lambda_{\alpha c} \cdot \Delta t \tag{1.13}$$

where we used (1.11) and the definition of $p_{\alpha c}$ given at the first step of the algorithm. Substituting in (1.12) we obtain

$$E_t\left[\sum_{k=1}^{N_\alpha(t)} S_\alpha\right] = \int_{\mathbb{R}} E_t\left[S_\alpha \,\middle|\, N_c = 1 = N_\alpha, \Pi^\alpha = \pi\right] \cdot \beta_\alpha(\pi) \cdot \mathrm{d}\pi \cdot \lambda_{\alpha c} \cdot \Delta t \,. \tag{1.14}$$

Now we can use the probability density of S_α

[6] Note that for the synchronous algorithm this is the only property of the Poisson distribution we use in this proof. The deduction can hence be generalized to every distribution with "finite speed", that is verifying the equations (1.11).

$$P_t\left[S_\alpha \in B_2 \,\middle|\, N_c = 1 = N_\alpha, \Pi^\alpha = \pi\right] = \int_{B_2} \beta_{11}(s;\pi)\cdot \mathrm{d}\, s \qquad \forall B_2 \in \mathcal{B}\left(\mathbb{R}^{n(\alpha)}\right) \tag{1.15}$$

where $\beta_{11}(s;\pi) := \beta_{11}(s;1,1,\pi)$, see (1.5) and the following notes. So that

$$E_t\left[S_\alpha \middle| N = 1 = N_\alpha, \Pi^\alpha = \pi\right] = \int_{\mathbb{R}} s\cdot \beta_{11}(s;\pi)\cdot \mathrm{d}\, s\,.$$

Substituting in (1.14) and in (1.7) we obtain the final result:

$$\begin{aligned} E_t\left(S_{t+\Delta t}\right) &= E_t\left(S_t\right) + \sum_{\alpha=7}^{10} E_t\left[\sum_{k=1}^{N_\alpha(t)} S_\alpha(t)\right] = \\ &= E_t\left(S_t\right) + \sum_{\alpha=7}^{10} \int_{\mathbb{R}} E_t\left[S_\alpha \middle| N_c = 1 = N_\alpha, \Pi^\alpha = \pi\right]\cdot \beta_\alpha(\pi)\cdot \mathrm{d}\,\pi \cdot \lambda_{\alpha c}\cdot \Delta t = \\ &= E_t\left(S_t\right) + \Delta t \cdot \sum_{\alpha=7}^{10} \lambda_{\alpha c}\cdot \int_{\mathbb{R}}\int_{\mathbb{R}} s\cdot \beta_{11}(s;\pi)\cdot \mathrm{d}\, s\cdot \beta_\alpha(\pi)\cdot \mathrm{d}\,\pi \end{aligned}$$

That is (making explicit both the dependence from time t and from the present value of the global configuration $e_t \in E = \mathbb{R}^D$)

$$S_t' = \sum_{\alpha=7}^{10} \lambda_{\alpha c}(t)\cdot \int_{\mathbb{R}^2} s\cdot \beta_{11}(s;\pi,e_t)\cdot \beta_\alpha(\pi,e_t)\cdot \mathrm{d}\, s\,\mathrm{d}\pi \tag{1.16}$$

This is the first RDE for the conditional expected value of the variable S_t in the given cell c. Note that the randomness is introduced by the global variable e_t.
Equation (1.16) cannot be solved alone, neither by numerical methods. In fact thinking e.g. to the Euler method we can read it as

$$E_t\left(S_{t+\Delta t}\right) = E_t\left(S_t\right) + \Delta t\cdot \sum_{\alpha=7}^{10} \lambda_{\alpha c}(t)\cdot \int_{\mathbb{R}^2} s\cdot \beta_{11}(s;\pi,e_t)\cdot \beta_\alpha(\pi,e_t)\cdot \mathrm{d}\, s\,\mathrm{d}\pi \tag{1.17}$$

So that if we start at $t = 0$ from a deterministic configuration e_0, then (1.17) gives the value of $E_0\left(S_{\Delta t}\right) = E\left(S_{\Delta t}\right)$; but this is only a very partial information about the distribution of $S_{\Delta t}$, and hence we cannot re-start the procedure even at the first step. This is the fundamental reason for which we need the RDE for all the moment of S_t at every time t.

The RDE for the other moment

Here we only sketch the final results. The first idea is to generalize the context (so as to obtain simplified notations). Generally speaking we have in fact a step relation (see (1.4)) of type

$$S_{t+\Delta t} = f\left[S_t, \Delta t, t, e_t\right] \tag{1.18}$$

To obtain the other equations for the moments it suffices to start from the k-th central moment

$$m_k\left(S_{t+\Delta t}\right) = E_t\left(S_{t+\Delta t} - E_t\left(S_{t+\Delta t}\right)\right)^k$$

and substituting $S_{t+\Delta t}$ and $E_t\left(S_{t+\Delta t}\right)$ with the relations (1.18) and (1.17) respectively it suffices to perform some algebraic calculation using as Δt a first order infinitesimal, so that $\left(\Delta t\right)^k = 0$.
The final equations for $k > 1$ are

$$\frac{dm_k\left(S_t\right)}{dt} = k \cdot E_t\left[\left(x_t - E_t S_t\right)^{k-1} \cdot \left(\partial_2 f - E_t \partial_2 f\right)\right],$$

where $\partial_2 f$ is the partial derivative of f w.r.t. the second variable Δt.
It is interesting to note that assuming f analytical, Markov processes and a first order infinitesimal standard deviation of S_t we can prove that the system of moment equations reduces to the ODE about the expected value $E\left(S_t\right)$ of the state variable deduced from the ME.

Now the above sketched Euler method can be intuitively completed: starting from an initial condition e_0 (and hence S_0) and an infinitesimal time interval dt, using the moment equations we can obtain the value of each moment $m_k\left(S_{t+dt}\right)$ at $t + dt$. If the moment problem can be uniquely solved, then we obtain the whole distribution of S_{t+dt}; using this distribution we can generate the next step S_{t+dt} and restart the procedure.
Even if we finally obtain an infinite system of differential equations, in our case it is natural to make assumptions on f such that the solution S_t of the RDE is bounded (every state variable is either a volume or a surface). Then it is possible to study the corresponding "truncated moment problem", keeping the amount of order up to a finite k and look for the convergence of the Euler procedure.

References:

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